Parity of Selmer ranks in quadratic twist families

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Families of quadratic twists

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This becomes isomorphic to E over $\mathbb{Q}(\sqrt{d})$ be is (in general) not isomorphic to E over $\mathbb{Q}$ and gives a convenient family of elliptic curves in which to ask statistical questions.

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Ordering twists: For each $X > 0$, let $\mathcal{C}(K,X)$ be the subgroup

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For each X, this is a finite subgroup of $\mathcal{C}(K)$.

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- Generalized to Jacobians of odd degree hyperelliptic curves by Yu.
- Predicted by (and consistent with) root number considerations.

General principally polarised abelian varieties

A/K principally polarised abelian variety (e.g. Jacobian).
We have

$$\dim_{\mathbb{F}_2} \mathrm{Sel}_2(A/K) = \dim_{\mathbb{F}_2} A(K)[2] + \mathrm{rk}_2(A/K) + \dim_{\mathbb{F}_2} Ш_{\mathrm{nd}}(A/K)[2].$$

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- How do parities of 2-Selmer ranks behave in quadratic twist families?

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If $K = \mathbb{Q}$ and $\dim A$ is odd then $\delta = 0$ but for $\dim A$ even this need not be the case (c.f. example of Yu.).

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Let A/K be a principally polarised abelian variety and let $\epsilon : \mathrm{Gal}(K(A[2])/K) \to \{\pm 1\}$ be the map $\sigma \mapsto (-1)^{\dim_{\mathbb{F}_2} A[2]^{\sigma}}$.

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- If we have $\text{Gal}\left(K(A[2])/K\right) \cong \text{Sp}_{2\dim A}(\mathbb{F}_2)$ and $\dim A \geq 2$, then again ϵ is *not* a homomorphism.

An example

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On the other hand, J has $\delta = \frac{3}{16}$ and odd 2^{∞}-Selmer rank, so

$$\mathrm{Prob}\left(\mathrm{rk}_2(J^{\chi}/\mathbb{Q}) \text{ is even}\right) = \frac{19}{32}.$$

Comparing 2-Selmer ranks modulo 2

Exploiting the isomorphism $A[2] \cong A^{\chi}[2]$ to view both $\mathrm{Sel}_2(A/K)$ and $\mathrm{Sel}_2(A^{\chi}/K)$ inside $H^1(K, A[2])$ yields

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Lemma

$$\dim_{\mathbb{F}_2} \mathrm{Sel}_2(A^{\chi}/K) \equiv \dim_{\mathbb{F}_2} \mathrm{Sel}_2(A/K) + \sum_{v \in \Sigma} \kappa_v(\chi_v)$$

$$+ \sum_{v \notin \Sigma,\ \chi \text{ ramified at } v} \dim_{\mathbb{F}_2} A[2]^{\mathrm{frob}_v} \pmod 2$$

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When $v \nmid 2\infty$ and A has good reduction at v then

$$\gamma_v(\chi_v) = \begin{cases} \dim_{\mathbb{F}_2} A[2]^{\mathrm{frob}_v} & \chi_v \text{ is ramified} \\ 0 & \text{otherwise.} \end{cases}$$

Thank you for your attention!